Pseudo-Unsteady Method for the Computation of Transonic Potential Flows

J. P. Veuillot* and H. Viviand†
Office National d'Etudes et de Recherches Aérospatiales, Châtillon, France

Abstract

A NEW pseudo-unsteady method has been developed for the computation of transonic two-dimensional steady inviscid flows. This method is based on a set of two equations, with artificial time derivative terms, derived from Euler's equations by assuming that the total enthalpy and entropy are constant throughout the flowfield, even during the transient state, so that only the steady-state solutions are of physical interest. This pseudo-unsteady system is hyperbolic; a stability criterion for explicit numerical schemes is determined from the knowledge of the characteristic cone. Discretization of this system is carried out directly in the physical plane for arbitrary curvilinear meshes by means of an extension of MacCormack's scheme. The domain of application is restricted to potential flows (but including weak shock waves). The main advantage of this method, by comparison with the widely used relaxation methods for the potential equation, is the simplicity of its implementation both in conservative form and for general flow configurations. Numerical applications have been carried out for transonic flows past cascades, in nozzles, and in channels.

Contents

Steady isoenergetic and homentropic two-dimensional (plane or axisymmetric) flows can be described by means of two variables such as the density ρ and the inclination angle θ of the velocity vector V with respect to a reference axis. The modulus V of the velocity and all the thermodynamic properties are then known functions of the density. The present pseudo-unsteady method is based on a set of two first-order unsteady-type equations for the two basic dependent variables ρ and θ, derived respectively from the continuity equation and the momentum equation. For numerical stability reasons to be discussed later, the unsteady term $\partial\rho/\partial t$ of the continuity equation is generalized in the form $\partial\zeta(\rho)/\partial t$, where $\zeta(\rho)$ is a given but yet unspecified function of ρ. Introducing the assumption that the total enthalpy and entropy are constant in the momentum equation written in Crocco's form, we get

$$\partial V/\partial t + \omega\Lambda V = 0 \tag{1}$$

where $\omega = \mathrm{rot}\, V$ is the vorticity, and we are left with the problem of deriving one scalar equation from Eq. (1).

If we consider the two components of Eq. (1) on Cartesian or cylindrical coordinate axes (respectively for plane or axisymmetric flow), there is no reason to single out either one. The choice to be made becomes obvious only if we write Eq. (1) in the local intrinsic axes with unit vectors s, n such that s is in the direction of V ($V = Vs$) and n is directly normal to s in the plane of the flow. Using the relations $\partial s/\partial t = n\partial\theta/\partial t$ and $\omega\Lambda V = \omega V n$ (with $\omega = \omega s \Lambda n$), we see that the s-component of Eq. (1) reduces to $\partial V/\partial t = 0$, which must be discarded, and that the n-component gives $\partial\theta/\partial t + \omega = 0$, which is retained. Hence the pseudo-unsteady system to be used is

$$\partial\zeta(\rho)/\partial t + \nabla\cdot(\rho V) = 0 \tag{2a}$$

$$\partial\theta/\partial t + \omega = 0 \tag{2b}$$

which must be complemented by the relation $V = V(\rho)$. For the numerical solution, we consider Cartesian (plane flow, $\epsilon = 0$) or cylindrical (axisymmetric flow, $\epsilon = 1$) x,y coordinates and write Eq. (2) in the following conservative form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\zeta(\rho)\\ \theta\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ v\end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ -u\end{pmatrix} + \epsilon\frac{v}{y}\begin{pmatrix}\rho\\ 0\end{pmatrix} = 0 \tag{3}$$

where $u = V\cos\theta$, $v = V\sin\theta$, and $V = V(\rho)$, now taking θ to be the angle of the velocity with respect to the x axis. At steady state, Eq. (2) or (3) reduces to the steady continuity equation and to the irrotationality condition, and the jump relations across a discontinuity line of the weak solutions correspond to the conservation of mass and of tangential velocity. Therefore, the steady-state solutions of Eq. (2), in conservative form, have a physical meaning as steady isentropic irrotational flows including weak shock waves.

Magnus and Yoshihara[1] have proposed a pseudo-unsteady approach based on a set of equations similar to Eq. (3), but with different unsteady terms given by $f = (-\tilde{u}, \tilde{v})^t$ where $\tilde{u}, \tilde{v}$ are the dimensionless components of the velocity perturbation. However, it can be shown that the corresponding system of equations is not always hyperbolic so that, when this occurs, the method will fail to converge. In the recent work of Essers,[2] the approach of Magnus and Yoshihara is also discussed, and a system of equations close, although not identical, to Eq. (3) is proposed.

It can be shown that the system (2) is hyperbolic if $\chi = d\zeta/d\rho$ is positive, which we assume to be the case. With a view to establishing a stability criterion for the numerical solution of this system by means of explicit schemes, we have determined the geometry of the characteristic cone in space-time at an arbitrary point $P_1(t_1, x_1, y_1)$. The trace of the cone in a plane $t = t_0$ is an ellipse with one axis, P_1x_1, parallel to $V(P_1)$.

Note that all the preceding results are independent of the particular law $V = V(\rho)$ used; in particular, we have not assumed constant specific heats. Typical configurations of this ellipse, for various Mach numbers, are shown in Fig. 1, for the case of a perfect gas with constant specific heats of ratio $\gamma = 1.4$ and for $\zeta(\rho) = -\rho_* V(\rho)/a_*$ (a is the sound speed and * refers to sonic conditions).

To determine an approximate stability criterion for explicit schemes, we make use of the geometrical interpretation of the C.F.L. rule, according to which the numerical domain of dependence must enclose the exact domain of dependence as

Presented as Paper 78-1150 at the AIAA 11th Fluid and Plasma Dynamics Conference, Seattle, Wash., July 10-12, 1978; synoptic submitted Sept. 26, 1978; revision received Feb. 21, 1979. Full paper available from AIAA Library, 750 Third Avenue, New York, N.Y., 10017. Price: Microfiche, $2.00; hard copy, $5.00. **Order must be accompanied by remittance.**
Index categories: Transonic Flow; Computational Methods.
*Research Scientist.
†Division Head, Theoretical Aerodynamics. Member AIAA.

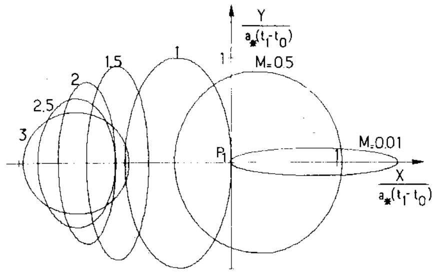

Fig. 1 Domains of dependence for various Mach numbers. $\zeta(\rho) = -\rho_* V/a_*$.

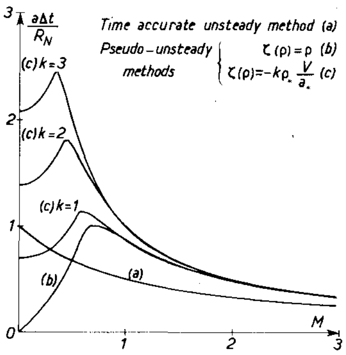

Fig. 2 Stability curves.

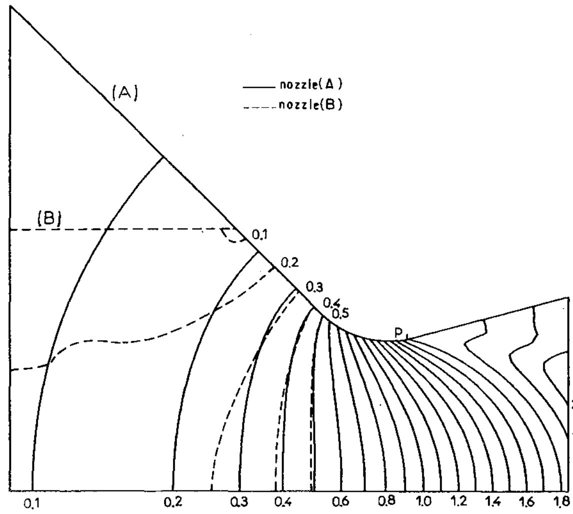

Fig. 3 Iso-Mach lines of transonic flows in nozzles.

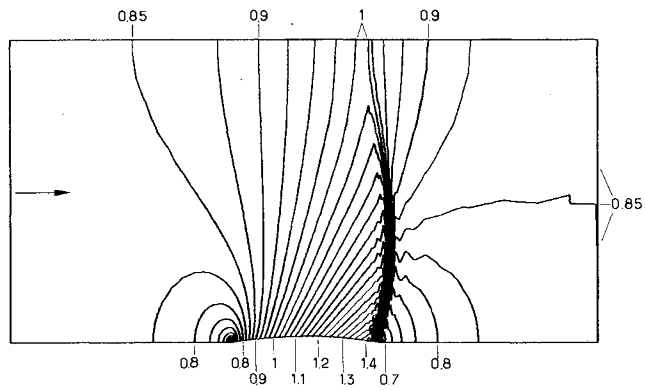

Fig. 4 Iso-Mach lines of transonic flow past a circular arc profile in a channel.

determined from the characteristic cone. For simplicity the numerical domain of dependence is taken to be a circle centered at P_l and of radius R_N (of the order of the mesh size). The exact domain of dependence for a time step Δt is an ellipse which is the intersection of the characteristic cone with the plane: $t = t_l - \Delta t$. By writing that this ellipse is inside of and just tangent to the circle of radius R_N, the maximum value of $a\Delta t/R_N$ is obtained as a complicated function of the Mach number M and χ not given here; we note only the result in the case of low Mach number, namely $a\Delta t/R_N \leq M\chi$, which shows that convergence difficulties are bound to occur in a low velocity region (for instance near a stagnation point) if $M\chi$ tends to zero with M (for instance with $\zeta = \rho$, i.e., $\chi = 1$). Therefore the function $\zeta(\rho)$ actually used was determined so as to avoid this difficulty, by requiring that $M\chi$ not vanish for $M = 0$ (and that ζ remain bounded). From the relation $d\zeta/dV = M\chi\rho/a$, we see that $d\zeta/dV$ must not vanish for $M = 0$ and the simplest choice satisfying this condition is

$$\zeta(\rho) = -k\rho_* V(\rho)/a_* \tag{4}$$

where k is a positive constant (in order for χ to be positive). Stability curves for different values of k (and for a perfect gas with $\gamma = 1.4$) are shown in Fig. 2 and compared with the stability curve (based on the same C.F.L. rule) of a time-accurate method using the exact Euler equations.

The basic pseudo-unsteady system of Eq. (3) is numerically integrated step by step in time by using an extension of MacCormack's two-step explicit scheme which allows discretization of Eq. (3) directly in the physical plane on an arbitrary mesh and without using a geometrical mapping. At the boundary mesh points, the original Eq. (2) is replaced by two compatibility relations corresponding to the two characteristic planes whose traces in a plane t = const are parallel to the boundary. The number of boundary conditions is discussed according to the type of the boundary: solid or permeable, upstream or downstream, subsonic or supersonic.

Various numerical applications have been carried out for internal flow problems (nozzles, cascades, ejectors) using the function $\zeta(\rho)$ given by Eq. (4) with $k = 1$. Numerical results are presented here for two typical applications (perfect gas, $\gamma = 1.4$). The first one corresponds to the shock-free transonic flow in axisymmetrical nozzles noted (A) and (B) on Fig. 3. Nozzle (A) is made of a 45 deg conical convergent and a 15 deg conical divergent with a circular meridian section of radius R, such that $R/L = 0.625$, where L is the throat radius. In nozzle (B), the 45 deg convergent has been truncated and partly replaced upstream by a cylindrical duct in order to test the stability properties of the method near a stagnation point. In the two cases, the direction of the flow is specified on the upstream subsonic boundary, but no condition is imposed on the downstream supersonic boundary. Fig 3 shows iso-Mach lines in nozzles (A) and (B). The second example corresponds to the transonic flow with a shock wave, past a circular arc profile with relative thickness 8.4% placed along the axis of a channel; the ratio of channel half-height over profile chord is 2.073. The direction of the flow is specified on the upstream subsonic boundary ($\theta = 0$) and the Mach number is given on the downstream subsonic boundary ($M = 0.85$). Fig. 4 shows iso-Mach lines.

Acknowledgment

The work reported herein was supported under DRET contract.

References

[1] Magnus, R. and Yoshihara, H., "Steady Inviscid Transonic Flows over Planar Airfoils. A Search for a Simplified Procedure," NASA CR-2186, Jan. 1973.

[2] Essers, J. A., "Time Dependent Methods for Mixed and Hybrid Steady Flows," von Karman Institute Lecture Series 1978-4 in Computational Fluid Dynamics, Rhode St. Genese, Belgium, March 1978.